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USING THE RELATIONAL PARADIGM: EFFECTS ON PUPILS' REASONING IN SOLVING ADDITIVE WORD PROBLEMS

Abstract: Pupils' difficulties in solving word problems continue to attract the attention of researchers. While researchers highlight the importance of relational reasoning and modelling, school curricula typically use short word problems to develop pupils' knowledge of arithmetic operations and calculation strategies. The Relational Paradigm attributes the leading role in mathematics learning to the relational thinking development. Using this perspective, we implemented a new approach to teaching additive word problem solving in primary school, encouraging relational thinking and modelling. We compared the overall results of additive word problems solved by Grade Two elementary pupils in the experimental group (N=216) and in the control group (N=196). Our data shows: a) on average, the experimental group performed significantly better in problem solving than the control group; b) in the control group, there was a considerable lack of success in solving problems that require relational thinking—there was no such effect in the experimental group.

Key words: Mathematics teaching and learning, problem-solving, reasoning development, Relational paradigm

1 CONTEXT AND FOCUS OF THE RESEARCH

Schoolchildren's difficulties in solving arithmetic word problems are well documented in the literature (e.g., Barrouillet & Camos, 2002) and confirmed by international studies such as the PISA (Artigue, 2011). These difficulties are usually described by incorrect choices of arithmetical operations or the inability to solve a problem. Many children have trouble with some types of arithmetic word problems up to Grade 6 of primary school (Xin, Wiles & Lin, 2008).

The mathematics curriculum for primary school in Quebec pays special attention to the development of pupils' problem-solving skills and their conceptual understanding of arithmetical operations. Teaching approaches that are currently used in early grades involve solving simple word problems to introduce addition and subtraction as mathematical operations. Further approaches involve solving more complex problem situations where pupils can apply and further develop their knowledge of addition and subtraction. Many teachers in the Quebec region experience difficulties in supporting pupils in the development of a deeper understanding of addition and subtraction (xxx, 2013). They also report that while solving problems, pupils often choose the incorrect arithmetical operation.

In an effort to meet the needs identified by researchers and practitioners, we proposed a study on arithmetic word problem solving in Grades one and two. The aim of the study draws upon Vasilii Davydov theory of developmental instruction (Davydov, 2008) to produce an important change in students' strategies in solving word problems. Our study included five facets.

First, we designed and implemented a training program to help teachers review their practices and develop a new approach—named the Equilibrated Development Approach (EDA)—which supposedly helps to develop deeper knowledge in solving additive word problems (xxx, 2014). We provide a short summary of the EDA later in this paper.

Second, based on our theoretical explorations, we identified important tenets and developed learning tasks that implement these tenets (xxx, 2017). We also discuss some of these tasks later in this paper.

Third, we continuously tested and adapted teaching materials and procedures based on our ongoing classroom observations and teachers' feedback. (The detailed description of this process is not in the scope of this paper.)

Fourth, we conducted multiple individual task-based interviews with pupils to better understand their thinking processes. This part of the study (xxx, 2015) analyses and describes grade-two pupils strategies before, during, and after the implementation of the EDA. In section 5, we provide some examples of students' strategies.

Fifth, we collected ongoing quantitative data characterizing pupils' performance in additive problem solving in experimental and control classrooms.

In this article, we present: the theoretical platform we developed; a short overview of the teaching approach implemented (EDA); and the quantitative analyses of pupils' performance in solving additive word problems. We also present two excerpts from individual interviews with students further illustrating the shift in students' mathematical thinking. Our data suggests the existence of a causal relations between the EDA implemented in classrooms and the particularities of pupils' performance in word problem solving.

2 DISTINGUSHIG TWO PARADIGMES IN ADDITIVE PROBLEM SOLVING

Given that word problems involving addition and subtraction traditionally mark the beginning of teaching problem solving and the use of arithmetical operations in school, special attention was paid to these types of word problems in research. Among multiple aspects studied, we choose to discuss arithmetical operations, semantic structures, and pupils' strategies.

According to research (Nesher, Greeno, & Riley, 1982; Riley et al., 1984; Vergnaud, 1982), the semantic structure of a word problem takes on a great part of the difficulty that learners experience. Several categories of semantic structures, for example Change problems (where a quantity was added or removed) or Combine problems (where two quantities constitute two parts of a whole), are well-known[1]. It has been shown that within each category, problems with an unknown final state or unknown total are the easiest to solve.

Furthermore, Pape (2003) uses the notion of *consistent* and *inconsistent* language in a problem to explain solving difficulties. Similarly, Bednarz and Janvier (1993) distinguish *connected* and *disconnected* problems. Pape's notion of *consistent language* and Bednarz and Janvier's notion of *connected* problems (e.g., *Peter had 3 marbles. He won 5 marbles. How many marbles does he have now?*) references problems where one can connect two known quantities (3 and 5 marbles) to a known action (won) which can be easily transformable into an arithmetical operation $3 + 5 =$. The terms *inconsistent language* or *disconnected problem* describe a problem, which requires the semantics of the story to be transformed into an operation having a different meaning. (*Peter had 3 marbles. He* ***won*** *some marbles. Now he has 8 marbles. How many marbles has he won?* Operation: $8 - 3 =$). This line of argumentation (about problem's classes and levels of difficulty) implicitly suggests that the understanding of the problem is based on the knowledge of key words and associated operations: won means add.

[1] For a detailed analysis of problems classifications, see (Carpenter et al., 1993; Nesher et al., 1982; Riley et al., 1984; Vergnaud, 1982).

Conversely, researchers studying upper-primary and secondary pupils' problem solving focus more on the mathematical structure of a problem and pupils' modelling activity rather than on the operations per se. Empirical research (e.g., Fagnant & Vlassis, 2013; Gamo, Sander & Richard, 2009; Ng & Lee, 2009; Xin et al., 2008) suggests that teaching methods and techniques that focus on pupils' attention to the mathematical structure of the problem, i.e. modelling activity, give rise to the solving strategies based on mathematical relationships and contribute to learners' improving their word problem solving skills.

A careful study of research literature in problem-solving and mathematical thinking development reveals the existence of two different paradigms: one drawing upon arithmetic operations and calculation strategies and the other putting forward quantitative relationships and modelling. The important distinction between the paradigms that we describe below informs the theoretical foundation of our work and, we believe, makes an important contribution to the field of mathematics education.

2.1 THE OPERATIONAL PARADIGM

The approach of analyzing problem-solving as the transformation of the wording of a problem into an arithmetical *operation* through the use of semantic structure is reflected in the work of many scholars (e.g., Bednarz & Janvier, 1993; Carpenter et al., 2008; Nesher et al., 1982; Riley et al., 1984). We have identified this approach as the Operational Paradigm. Within the Operational Paradigm, one sees arithmetical operations as the most important mathematical knowledge to be developed and used when analyzing problems. One also considers a word problem as numerical data connected by a semantic link in a story to be transformed into numbers connected by an operation (Figure 1).

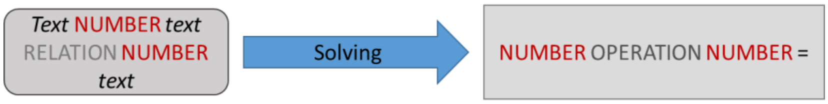

Figure 1 Solving process in the Operational Paradigm

We have identified several facets of the Operational Paradigm. For example, word problems are defined by the arithmetical operation they involve, i.e., *addition* or *subtraction problems*. Since word problems are considered as tools for learning operations on numbers, there is no distinction between a *solution* strategy and a *calculation* strategy when solving a problem. For example, the counting forward calculation strategy for the second marbles problem $(3 + ? = 8)$ can be considered an appropriate *solution* strategy (Carpenter et al., 2008).

The development of problem-solving abilities within the Operational Paradigm can be generalized as follows; at the first stage, pupils can see the problem as a story, mimic it using objects, and construct the answer as the final state of this mimicking (Nesher et al., 1982; Riley et al., 1984). Pupils reason in a sequential way, seeing the problem as a process and as a model of an addition or subtraction operation. During the final stage of development, pupils reason about the problem in a holistic flexible way, seeing the problem as a relationship (Lesh & Zawojewski, 2007). They can transform the semantic relationship given in the story into a part-whole relationship and find the necessary arithmetical operation. The ultimate proof of the final stage of development is that the pupil "is able to read the word 'more', and yet perform a subtraction operation" (Nesher, 1982 p. 392). Thus, the most difficult problems for pupils are those expressed in inconsistent language (disconnected) requiring an inversion of their semantic structure.

Even though researchers recognize the importance of understanding semantic and quantitative relationships to solve a problem, they consider this understanding to be developed on the basis of arithmetical operations (rather than relationships). Thus, the Operational Paradigm views the development of thinking related to problem solving as going from sequential thinking (story/process/operation) to holistic flexible thinking (structure/object/system) (Figure 2).

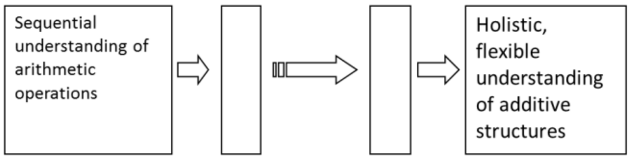

Figure 2 Thinking development in the Operational Paradigm

Several studies (e.g., Malara & Navarra, 2002; Thompson, 1993) acknowledged the limitations of this approach and discuss its possible negative effects on pupils' success in solving arithmetic word problems and on their prospective learning of algebra. Thompson (1993) studied problem-solving difficulties in upper-primary school. He explained:

> [Pupils' difficulties] stemmed from conceptualizing the dual roles played by quantities that calculations were meant to evaluate. To focus on calculational matters would distract children from addressing the sources of their difficulties. Second, children's school experience already orients them to reference-less operations on numbers. I suspect that an even greater emphasis on calculational subtleties would produce even greater distractions regarding children's conceptualizations of relationally-rich situations (p. 202).

Malara and Navarra (2002) bring forth similar arguments. They hypothesized that:

> The main cognitive obstacles in learning algebra are to be found in the pre-algebraic field, and that many of these spring up from unsuspected arithmetical contexts and they then become conceptual obstacles to the development of algebraic thinking, because of the weak conceptual control which many pupils have over the meanings of algebraic objects and processes.(p. 1)

To address these cognitive obstacles, they propose “teaching the pupil to think of arithmetic in algebraic terms” (p. 2) at early grades. This suggests a change of theoretical frameworks for research as well as for teaching practice.

2.2 THE RELATIONAL PARADIGM

Davydov (1982) proposes an alternative view on how pupils can develop their knowledge of arithmetic problem solving. He advances the notion that additive relationship, which is “the law of composition by which the relation between two elements determines a unique third element as a function” (Davydov, 1982, p. 229), is the basis of knowledge about addition and subtraction operations. This additive relationship is so fundamental that he insists on teaching this relationship prior to counting. We identify this approach as the Relational Paradigm.

Within the Relational Paradigm, the solver needs to understand the underlying additive relationship and only then can identify the arithmetical operation to calculate the unknown element of this relationship (Figure 3). One should name arithmetical problems according to the quantitative relationship they present. *Additive* word problems present additive relationships and they can be solved using addition or subtraction operations. In practical teaching and learning, there is an important distinction between *solution strategy* and *calculation strategy* when solving a problem. For example, in the case of the second marbles problem, the recognition of 8 marbles as a total and 3 marbles as a part yields a *solution* strategy $8 - 3 = ?$—standard mathematical expression—as finding the other part. This strategy is logically deduced from the identified relationship, it is independent from the numerical values involved and can be carried out using different calculation tools. The counting up strategy— $3 + ? = 8$ nonstandard expression—would be an appropriate *calculation* strategy for some particular numerical values and might be carried out mentally or with tokens.

However, this strategy as a solving strategy might not reflect the holistic and flexible understanding of the relationship by the solver and it will be cumbersome with big or real numbers.

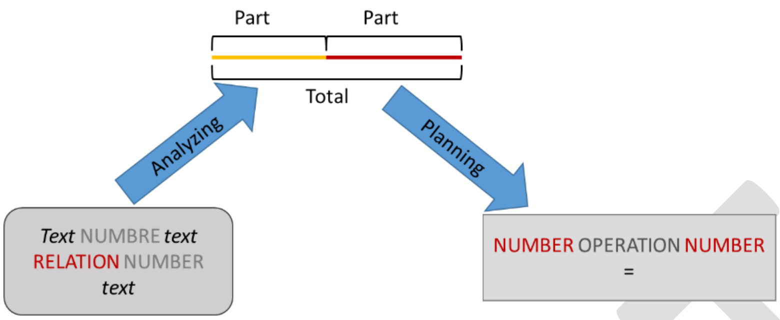

Figure 3 Solving process in the Relational Paradigm

Davydov (1982) proposed to begin the process of reasoning development by discussing with pupils the various relationships between physical objects, such as direct comparison or composition of amount of water in containers, paper plane figures (areas) or ropes (lengths). Within the Relational Paradigm, we can summarize the mathematical thinking development as moving from the understanding of additive relationships between physical objects (without numbers) toward a holistic and flexible understanding of additive structures present in word problems (with numbers) (Figure 4).

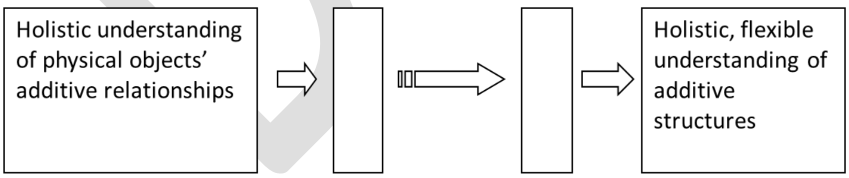

Figure 4 Thinking development in the Relational Paradigm

The two research paradigms—the Operational and the Relational—favour different modes of mathematical thinking **development** in problem solving. Within the Operational Paradigm, sequential thinking is used first. Within the Relational Paradigm, holistic (simultaneous) thinking is what drives things forward.

Within the Relational Paradigm, we can reinterpret the difficulties in solving word problems in the following way: pupils have difficulty solving additive word problems when they fail to recognize and operate on the additive relationship involved. Thus, the understanding of additive relationships plays the main role in additive problem solving. At the same time, we cannot deny the role that the literal, sequential understanding of a word problem plays because it gives solvers access to the problem-solving task through its interpretation within a real-life context. In the same vein, the understanding of addition and subtraction as processes (adding and removing) ensures the appropriate caring of these arithmetical operations.

3 THEORETICAL FRAMEWORK SUPPORTING EQUILIBRATED REASONING DEVELOPMENT

Inspired by Sfard's (1991) idea about the possible cognitive duality of mathematical thinking, we suggested that sequential thinking (addition and subtraction as processes) and systemic (holistic) thinking (additive relationship between three quantities as an object) can be conceived as a manifestation of this very duality of mathematical thinking, that is, as two distinct ways to think about a word problem. Sfard (1991) argues that the *object* and *process* ways of thinking, "although ostensibly incompatible, are in fact complementary" (p. 1).

We hypothesized that teaching approaches implemented within the Operational Paradigm promote pupils' natural thinking preferences—*sequential thinking* in particular. This can lead many pupils to an overdevelopment of or overreliance on sequential thinking in the context of problem solving and can at the same time inhibit the development of a systemic relational understanding of problems presented as stories. Such ingrained sequential thinking, thus, creates a disequilibrium and an inability to coordinate two different ways of thinking within the same problem-solving task. The traces of such disequilibrium should be observable in pupils' problem-solving performance. While developing sequential thinking, pupils should become increasingly successful in solving connected problems with consistent language (Bednarz & Janvier, 1993; Pape, 2003), but not in solving *difficult* problems, which require an inversion of their semantic structure. Steffe and Johnson (1971) studied pupils' arithmetic abilities and problem solving, and reported visible differences in how successfully Grade-one pupils solved additive word problems. They wrote:

> The mean scores for the two problem types a + b: A and a + b: N (A denotes described action; N denotes no described action.) were appreciably greater (no significance test performed) than the mean scores of the remaining problem types, which were all quite comparable (p. 58).

If both sequential and systemic thinking are being developed in coherence, pupils' reasoning will be better coordinated, and they will have similar success with all types of problems. Thus, our research question is: While implementing a developmental approach within the Relational Paradigm in the context of Grade-one and Grade-two of primary school, can we observe a significant difference in pupils' performance in terms of the equilibrium between sequential and systemic relational thinking?

Below, we provide the theoretical overview of the Equilibrated Development Approach (EDA) and a brief description of the Ethno-Mathematical model of problem solving we used to implement the teaching of additive word problems within the Relational Paradigm.

3.1 THE EQUILIBRATED DEVELOPMENT APPROACH

Based on the *reasoning duality* hypothesis described above, we designed and implemented a new approach, the Equilibrated Development Approach (EDA). This approach is used to teach additive word problem solving by supporting the simultaneous and equilibrated development of the two ways of thinking (the relational and the sequential).

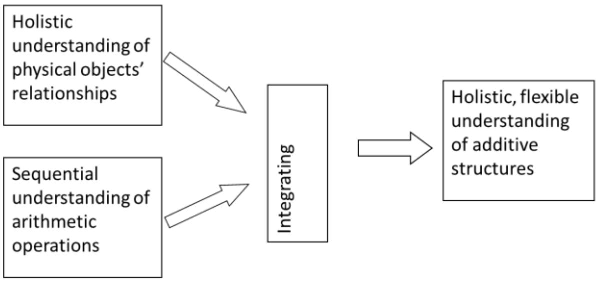

Figure 5 Thinking development in the Equilibrated Development Approach.

Within the EDA, the holistic understanding of relationships between physical objects (such as comparison or combination of strings of different lengths) and the sequential understanding of arithmetical operations make up the foundation of problem-solving knowledge development (Figure 5). Following the main idea of the Relational Paradigm, we advocate that word problems might be tools for relational thinking development. Therefore, the EDA engages pupils in an explicit analysis of the mathematical structure of a problem as a system of quantitative relationships. We used line segment diagrams (named Arrange-All diagrams) with reference to strings, to model these relationships and support the analysis of the mathematical structure of the problem. The distinguishing property of the Arrange-All diagrams is that they allow the representation of relationships without necessarily representing numerical values. Below, we used the Arrange-All method to represent the second marbles problem as an additive relationship (Figure 6).

Peter had 3 marbles. He won some marbles. Now Peter has 8 marbles. How many marbles did he win?

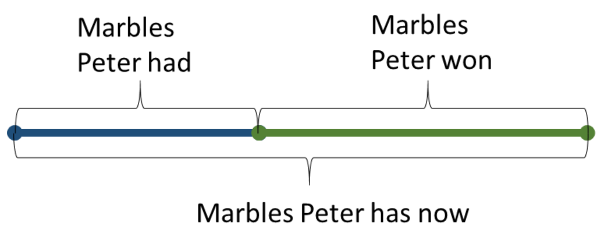

Figure 6 Arrange-All diagram for the problem. The marbles Peter had and the marbles Peter won compose the total of the marbles Peter has now.

3.2 THE ETHNO-MATHEMATICAL MODEL

To support the implementation of the EDA in the classroom, we used the Ethno-Mathematical model of the problem-solving process (Mukhopadhyay & Greer, 2001; xxx, 2015; xxx, 2008). We adjusted this model to support the relational reasoning in solvers. According to this model, to solve a problem, the solver should access the sociocultural context through the text of the problem (literally understand the story). Then, the solver should express her understanding of the situation within the mathematical context by creating a graphical model (ex. Arrange-All diagram), producing a holistic view of the quantitative relationships involved. In respect to the relational paradigm, we ask pupils to model relationships, not operations, numbers or objects. From this model, the necessary arithmetical operation can be derived, thus transforming the holistic view into a calculation sequence. For example, looking at the model in Figure 6, one can propose that in order to find "marbles Peter won," we need to remove/subtract "marbles Peter had" from "marbles Peter has now", or $8 - 3$. This deduction is highly general and supports solving problems that involve different types of numbers. The calculation step can also be modelled separately and carried out using different tools depending on the nature of numbers involved. The learner should then make sense of the calculation results in terms of the sociocultural context, evaluating it in relation to the initial understanding of the problem (Peter won 5 marbles, which is socioculturally reasonable within the given story). Thus, the problem-solving process is organized in a cycle, potentially supporting the development of both sequential and systemic (holistic) thinking in learners.

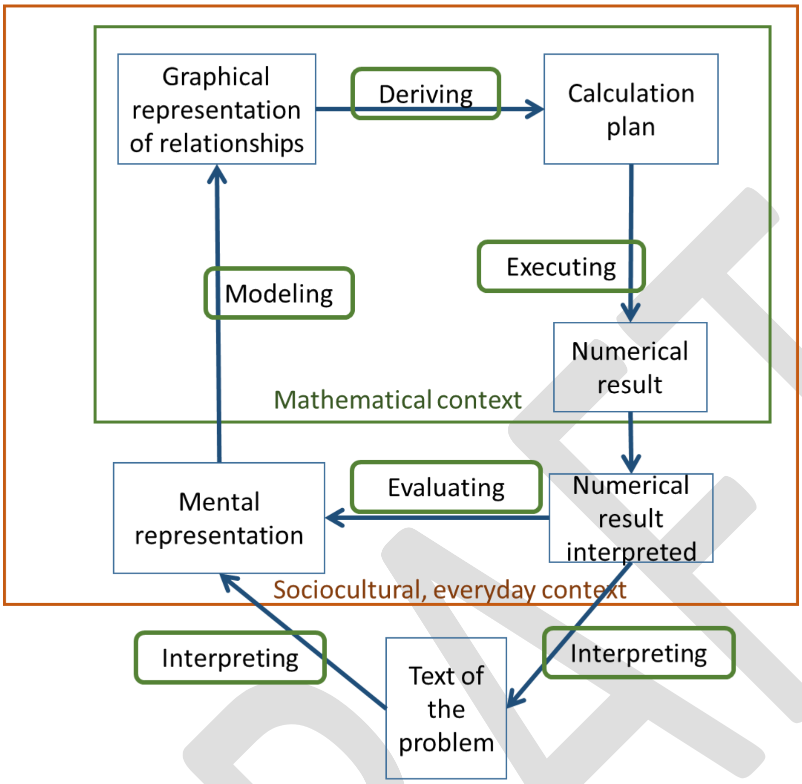

Figure 7 Ethno-Mathematical model of the problem-solving.

The Ethno-Mathematical model (EM model) is similar to the approach developed within the theory of Realistic Mathematics Education (REM) (Gravemeijer, Lehrer, Oers, & Verschaffel, 2002) which suggests building student mathematical thinking from real-life knowledge via modelling. However, the EM is different from the RME in two important details of its implementation. First, the students need to read the text prior to get to the “reality” of the situation. The reading directly affects the success of problem solving especially for very young students. Therefore, the EM model describes the reading and literal comprehension step explicitly. Second, consistent with the relational paradigm, EM states explicitly what should be modelled by pupil is the relationship(s) and not numbers, objects or operations. RME model is not clear about what pupil should model, thus it might allow pupil to model calculation instead of the relationships among the data. Summarizing the above differences, we suggest that the EM model is similar to the RME model, but more accurate in distinguishing between modelling (relationships) and calculation (operations and numbers).

4 METHODOLOGY

4.1 PARTICIPANTS

The study was conducted in French. Our participants came from both rural and urban schools. Our participants were teachers and their pupils who volunteered to participate were placed in either the experimental or the control group. All of the teachers had at least 5 years of teaching experience. During the three years of the study, 12 teachers participated in the professional development program that focused on additive problem solving for at least one year. The teachers implemented newly designed activities within the EDA in their 14 Grade-1 and Grade-2 classrooms (N=216 pupils, 6-8 years old, 101 girls and 115 boys). Fourteen other teachers agreed to participate as a control group (N=196 pupils, 7–8 years old, 81 girls and 115 boys). They did not participate in the professional development group and knew nothing about experimental activities and the EDA approach.

The majority of pupils were native French speakers (N=195 in the experimental group, N=160 in the control group). All teachers and the parents of the 412 pupils signed an ethical consent form. Within the experimental group, the majority of pupils (N=178) experienced traditional teaching in Grade-one and were taught using EDA over one school year in Grade-two. Some other pupils (N=13) experienced the EDA for one school year in Grade-one.[2] Other pupils (N=25) were taught using EDA in both Grades 1 and 2. Within the control group, all pupils experienced traditional[3] teaching in both Grade-one and Grade-two.

4.2 APPROACH TO PROBLEM-SOLVING IN THE EXPERIMENTAL GROUP

The teacher-participants integrated the EDA into their regular curriculum for teaching additive problem solving. None of them added extra time to work on mathematics with pupils, to train them to perform mental calculation, or to provide them homework on problem-solving.

A number of activities were implemented in class, during which the teachers invited pupils to analyze different situations involving the additive relationship between three quantities. For example, they presented pupils regularly with mathematically impossible situations such as: *Peter had 3 marbles. He then won 6 marbles. Peter now has 8 marbles*. Pupils constructed an Arrange-All diagram to represent the described additive relationship. They then used this model to find values (to replace 3, 6, or 8) to make the situation correct. We explain the use of this type of task in a classroom in a previous publication (xxx, 2017).

Teachers consistently used Arrange-All diagrams with each problem-solving task to support the dialogue around the additive relationships. They challenged pupils' understanding of the connection between the text of the problem and the diagram. They insisted that the mathematical expressions be written in a standard form ("$8 - 3 = ?$" being acceptable and "$3 + ? = 8$" being unacceptable).

[2] We tested the performance of these pupils at the end of Grade-one.

[33] Traditional teaching refers to the approach promoted by the schoolboard for the last several years.

Meaning that students should not only translate a sequence of events as “something was added”, but show that they understand the part-part-whole relationship of the situation in a flexible way. Further, students can *deduce* that a part can be found as a result of subtraction “whole-remove-part”. This element of deduction is critical to the distinction between sequential and holistic thinking.

The limitations of this paper do not allow for a detailed description of all particular tasks that were used in the project. However, the key point is that all these tasks and classroom activities attracted pupils’ attention to the additive relationship, by explicitly separating the analysis of the problem from the calculation, and therefore implementing the Relational Paradigm.

4.3 APPROACH TO PROBLEM-SOLVING IN THE CONTROL GROUP

Teachers in the control group followed the teaching instructions adopted by their schoolboard. According to the schoolboard consultant responsible for the math teaching in elementary schools, usual problem-solving practice includes: carefully reading the problem, identifying the numerical data, representing it using tokens or ten blocks (drawing) to support the calculation, and writing a mathematical expression in any form.

4.4 QUANTITATIVE DATA COLLECTION

The main results that we discuss here come from the written additive word problem solving tests (paper test) administrated to pupils at the end of their second year of schooling (7–8 years).[4] We collected these data during all three years of study, therefore the experimental pupils’ results come

[4] The 13 pupils who experienced the EDA during their first year sat the test at the end of the first year.

from different stages of the teachers' appropriation of the EDA approach. All participants completed the paper test.

During the first year of the study we also conducted a computerized test asking pupils to solve problems in more rigorous conditions (we describe below): 26 pupils from the experimental group and 98 pupils from the control group participated. Seeing as we could not provide the appropriate computer environment for all participants, we used small-scale computer tests to verify the reliability of the paper test conditions.

Given that the study was conducted in different classrooms, we did not control all specific variables of the teaching implemented by each teacher nor was it our intention. This gave us a larger view of the observed phenomena while limiting the control of some didactic variables.

4.5 QUALITATIVE DATA COLLECTION

During the first year of the study, the first author also conducted four sessions of individual semi structured problem-based interviews with 12 students from the experimental group (48 interviews). We used word problems similar to those of the written test to dig deeper into the students' reasoning and to witness the shift towards relational thinking in the students' problem-solving strategies (see a very detailed analysis in xxx, 2015).

4.6 WORD PROBLEM-SOLVING TESTS

Each test, whether paper or computer, included four additive word problems. Each problem had one additive relationship and included two numbers smaller than 50, and one extra (irrelevant) piece of data. We used the extra data to minimize the chance of pupils obtaining a correct answer by simply adding all the numbers together. The full text of the problems can be found in Annex 1.

Table 1 presents the problems' semantic structures according to the well-known classifications (Riley et al., 1984) and correct solutions.

Table 1 Problems' semantic structures according to Riley et al. (1984)

Problem	Structure	Correct solution
Ladybug	Combine (2) or Part-Whole, part unknown	17 - 8
Rolls	Change (3) or Positive change, change unknown	36 - 28
Homework	Change (6) or Negative change, total unknown	15 + 7 or 7 + 15
Marbles	Combine (1) or Part-whole, whole unknown	14 + 27 or 27 + 14
MarblesCOMP	Combine (2) or Part-Whole, part unknown	43 - 27
HomeworkCOMP	Change (4) or Negative change, change unknown	21 - 7
RollsCOMP	Change (1) or Positive change, result unknown	28 + 8 or 8 + 28
LadybugCOMP	Combine (1) or Part-Whole, whole unknown	8 + 9 or 9 + 8

Before starting the test, we explained to the pupils that the desired solution is the mathematical expression, which explains how to calculate the answer to the problem. We presented them two example problems that were solved on the blackboard and we put the answers both in the acceptable form $2 + 3 = \square$ and unacceptable form $2 + \square = 5$, we then crossed out the unacceptable form on the blackboard. We say: "the operation you propose as a solution should not contain an unknown, It should appear as a result". We also allowed pupils to use tokens, drawings, and other techniques they usually used in class. However, we insisted that they give a mathematical expression, emphasizing that the final number (the answer) was not so important. Very few pupils used tokens. The pupils were allowed to request reading assistance, which was provided by the researcher, and they were permitted to ask questions in case they did not understand words or expressions used in the word problem. Pupils were given an unlimited time to complete the test.

We considered the problem as successfully solved if the pupil provided the correct mathematical expression in the expected standard form. In line with the Relational Paradigm, mathematical expressions containing an unknown as a term of the operation can describe the situation, but not the arithmetical operation to be carried out. The transformation of such expressions into the standard form requires holistic thinking and showcases the difficulty of inversion studied in this project. In absence of the correct mathematical expression, even if the numerical answer was correct, we considered the problem unsolved. Being able to communicate a solution strategy using mathematical expressions represents a different type of thinking than just mentally calculating the correct number.

Within the computer environment, the solver needed to compose a mathematical expression by dragging data from the text and an operation into the reserved space. There was no possibility to compose an expression in a nonstandard form, because no "?" sign (or other) was available. Therefore, the computer test presented more rigorous conditions as it excluded the possibility for pupils to follow the classroom "tradition" of using nonstandard expressions. For both groups of pupils, it was the first time they had used this type of computer environment. We used this test with only 26 experimental pupils, because the other experimental pupils had used the computer environment within other learning activities. Therefore, the data from them would have been biased.

4.7 REASONING EVALUATION MEASURES

We coded the successful solving of a particular problem by a pupil as “1” and failure to do so as “0.” In order to characterize pupils’ reasoning in problem solving, we chose two measures. The first measure characterized the *average success* (AS) of a pupil in solving all four problems on a test: AS=1 meant that all problems were solved, and AS=0 meant that none of the problems were solved.

The second measure characterized the difference between success in solving *easy* problems and *difficult* problems, thus showing pupils’ sensitivity to *difficult/easy* problems. In order to determine which of the four problems were *easy* and which ones were *difficult*, we adopted the idea of Steffe and Johnson (1971), who based their judgment about a problem’s difficulty by the mean values of success for each problem. We then compared the control group’s success in solving the test problems. Table 2 presents the results.

Table 2 Descriptive Statistics, control group

	N Statistic	Mean Statistic	Std. Deviation Statistic	Skewness Statistic	Std. Error
Ladybug	196	.33	.47	.75	.17
Rolls	196	.34	.48	.67	.17
Homework	196	.67	.47	-.72	.17
Marbles	196	.59	.49	-.38	.17
MarblesCOMP	98	.17	.38	1.75	.24
HomeworkCOMP	98	.17	.38	1.75	.24
RollsCOMP	98	.32	.47	.75	.24
LadybagCOMP	98	.47	.50	.13	.24

It is quite clear that on the paper test, pupils had more success with the Homework and Marbles problems than with Ladybug and Rolls problems. On the computer test, the MarblesCOMP and HomeworkCOMP problems are shown as being more difficult than the RollsCOMP and the LadybagCOMP. This corroborates with Steffe and Johnson’s observations.

Thus, we defined the *difference factor* (DF) for a pupil for the paper test as $DF = (Homework + Marbles - Ladybug - Rolls)/2$. The value of $DF = 1$ meant that the pupil solved the two easy problems and did not solve the two difficult ones. The value of $DF = -1$ meant that the pupil solved the two difficult problems and none of the easy ones. The value of $DF = 0$ means that the pupil solved as many easy problems as they did difficult ones.

For the computer test, $DF = (LadybagCOMP + RollsCOMP - MarblesCOMP - HomeworkCOMP)/2$.

We calculated these two measures (AV and DF) for each pupil in all groups for both paper and computer conditions. We also calculated each group's average success (GAS) as the mean value of the AS in each sample. The group difference factor (GDF) was calculated as the mean value of DF in each group including only the pupils who solved at least one problem, but not all of them (complete failure and total success data can bias the differentiation calculation).

5 RESULTS

We obtained the following results for the average success in solving problems within the two conditions. Table 3 presents the semantic structure of each problem, pupils success, and corresponding results obtained by Steffe and Johnson (1971).

Table 3 Pupils' performance proportion for paper and computer problems compared to Steffe and Johnson (1971, p. 59)

Structure	Experimental group success Paper N= 216 (Computer N=26)		Control group success N=196 (Computer N=98)		Steffe and Johnson (1971, p. 59) Grade-one pupils N=125 success (No objects present)
Combine (2)	0.65	(0.38)	0.33	(0.17)	(a + x: N) 0.38
Change (3)	0.56	(----)	0.34	(-----)	(a + x: A) 0. 41

Change (4)	------	(0.46)	-------	(0.17)	(a + x: A) 0.41
Change (6)	0.70	(-----)	0.67	(------)	(a + b: A) 0.70
Change (1)	------	(0.34)	-----	(0.32)	(a + b: A) 0.70
Combine (1)	0.59	(0.69)	0.59	(0.46)	(a + b: N) 0. 67
Mean score (GAS)	0.62	(0.47)	0.48	(0.28)	
Difference factor (GDF)	0.04	(0.13)	0.30	(0.42)	

We conducted a further statistical examination of each of the two characteristics of control and experimental populations (T-test using IBM SPSS software). The overall results of the paper test show that the mean score is significantly higher in the experimental group than in the control group $(.62 > .48 \;\; p < .01)$ with a moderate effect size d_{Cohen}= -0.437 (Confidence Coefficient=95%). At the same time, the difference factor in the experimental group is closer to 0 than in the control group $(.04 < .30 \;\; p < .05)$ with a high effect size d_{Cohen}=0.691 (Confidence Coefficient=95%).

5.1 ANALYSIS OF QUANTITATIVE RESULTS

As we stated previously, the data obtained in the control group follows the pattern reported by Steffe and Johnson (1971). We can clearly see the difference in success between *difficult* (Combine 2, Change 3, 4) and *easy* problems (Change 6, Combine 1). The data obtained in the experimental group does not show the same pattern: the success are close within the paper test environment (0.56 - 0.70). Within the computer test environment, the success diverge further, but not for the same semantic structures as are in the control group, with Combine 2 and Change 1 problems being more difficult than Combine 1 and Change 4.

The evaluation of the second characteristic, the difference factor, may indicate the cause of the experimental group's higher performance in our study. The gap in performance between *difficult* and *easy* problems is significantly narrower in the experimental group than in the control group. As the average performance is not close to 0 or to 1.0, the narrow performance gap for different types of problems cannot be associated with the absence of knowledge or with full knowledge development.

We can only explain the small difference factor by the experimental group's equal success in solving difficult and easy problems.

Looking closer at the paper-test results, the following two figures show that the AS in the control group is distributed close to the normal. In the experimental group, the shift towards the maximum value is clearly visible.

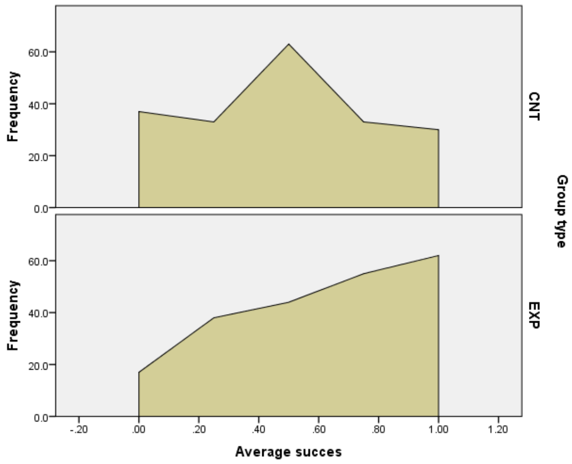

Figure 8 Average success in experimental and control groups

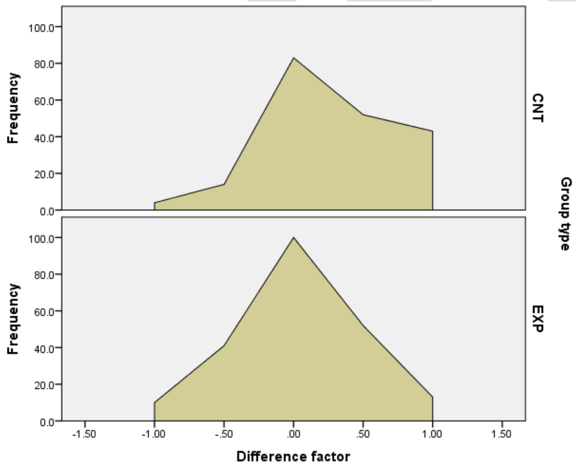

Figure 9 Difference factor in experimental and control groups

The distribution of the DF in the control group is shifted towards the maximum value—the greatest difference between easy and difficult problems. In the experimental group, the DF is distributed close to normal with the majority of cases near the middle value (DF=0).

According to our theoretical framework, relatively difficult problems require holistic flexible thinking for their solution and easier problems can be solved by using sequential thinking. The pupils' performance in solving additive problems in our study can be interpreted as follows. The pupils in the control group developed (or relied on) more sequential thinking than holistic thinking while being in the middle of their knowledge development process (GAS 48%). The pupils in the experimental group developed their mathematical thinking in better equilibrium or relied more on the relational holistic thinking. Holistic thinking allowed the pupils to be as effective in solving difficult problems as easy problems even though they were also in the middle of their knowledge development process (GAS 62%).

This conclusion is also supported by the data obtained with the computer test. Even though the experimental group showed an important variability in performance of the four computer problems, this variation does not correspond to the pattern observed in the control group. The control group pattern is the same as observed by many researchers who associated it with the semantic structure of problems. We can propose that the difficulties pupils in the experimental group had with RollsCOMP, MarblesCOMP, and HomeworkCOMP were not directly due to the problems' semantic structure. In a situation where the semantic structure is not the cause of the difficulty, other factors, such as context or wording start to play important roles producing different performance patterns.

Furthermore, the difference between sequential and holistic reasoning in solving word problems clearly appears through the comparison of students' solving strategies before and after the

EDA training. The following section reveals the difference in one student's thinking before training and after six months of in class EDA training.

5.2 SOME QUALITATIVE DIFFERENCE IN STUDENT'S THINKING

We present here two excerpts from interviews (translated from French) conducted at the beginning and at the end of the year when the EDA was implemented in a Grade-Two classroom for six months. Both interviews involve the same student, Eva, who performs at grade in mathematics according to her teacher.

Before training. Eva solved the following problem: *There were 34 logs in the bag that the father bought to make a campfire. The fire is lit for 48 minutes. Some logs are already burnt. There are 27 logs left in the bag. How many logs were burnt?* The experimenter read the problem to the student.

> Experimenter: Explain it to me, please. Can you explain?
>
> Eva: I look first to the digits. Like, there is 34 logs, there are some burnt, then there are 27 that are… that are burnt. Then I look, there is 48, I think. I don't remember.
>
> …
>
> Eva: [reads the problem by herself] Now, I think I understand, cause, there are 34 logs. Then, there are 27. There are 27 left. In my head, I know how to remove them, the ones that burnt. Can I do it?

In this episode, Eva demonstrates that she was paying great attention to the numbers in the problem.

> Eva: [draws 34 circles and starts to cross them out one by one from the end. Stops after crossing 5 circles. She recounts remaining circles to finally adjust her representation to the story.]

In this episode, the student goes back and forth to adjust her representation of the story.

Experimenter: Can you answer the question now?

Eva: Yes. There are left… burnt, there are 1, 2, 3, …[counts silently crossed circles]. There are 7 burnt.

In this episode, Eva uses the *mimicking* strategy—she draws the events of the problem *sequentially* to figure out the final state and counts the burnt logs. Somehow, Eva modelled the situation and figured out the correct numerical answer. However, she did not analyze the part-part-whole relationship and did not deduce the arithmetic operation in a general way.

Figure 10 Eva uses mimicking strategy

After training (in 6 months). Eva solved the following problem: *In 2011, there were 365 days. In her calendar, Julie marked 198 school days and 10 holidays. How many days were without school in 2011?* The student reads the problem and the experimenter discusses with the student the meaning of the expressions: *school days*, *holidays*, *year 2011*.

Experimenter: What can we do?

Eva: I think that I need to do this because I understood… [Draws a horizontal line] I do not understand but I think we need to do this. [Put arcs above and under the horizontal line in a way similar to the part-part-whole diagram].

Eva: [answering experimenter’s questions about her drawing] In total, there is this [encircles the number 365 in the text and put it on the diagram].

Eva: In total is what is all-together [making a gesture *total* with her two hands].

Eva: It is vacations and school days.

In this episode, we see Eva respecting the learned process of solving a problem i.e. drawing a diagram first. She pays attention to the quantity and its role within the relationship (total, all days).

Experimenter: OK. [Shows the two above arcs of the diagram] Here, you also represented something? What is it?

Eva: I think I need to put this (encircle the number 198 in the text and put it over the right arc of the diagram].

Experimenter: What is it?

Eva: Schooldays, and here [points with her finger to the left part of the diagram] is what we are looking for. In the… [points with her finger to the question of the problem] This is…, we are looking for this because we do not know what it is.

In this episode, Eva correctly identifies parts of the whole by linking the elements of the diagram to the problem's data. Next, she produces a correct calculation expression $365 - 198 =$.

Experimenter: And if I ask you, should we add?

Eva: I don't know.

Experimenter: You do not know?

Eva: No. I think it should be minus.

Experimenter: You think it is minus. Why do you think it is minus?

Eva: [points to the right part of the diagram which is 198]. If we remove what is at school, we will know what is here [points to the question of the problem].

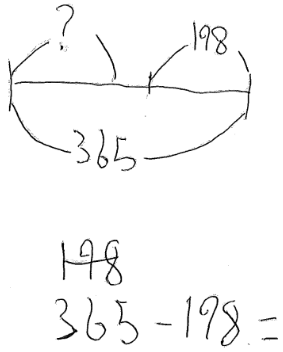

Figure 11 Eva uses diagram strategy

In this last part of the dialogue, the experimenter challenged the strength of Eva's conviction. In response, Eva used her diagram as a model to justify her solution, thus demonstrating holistic relational reasoning.

Comparing these two episodes shows that the student's understanding of the reality of the problem does not directly produce the most general solution. It also contrasts sequential thinking and holistic thinking. Indeed, in both cases the student used strategies based on the reality of the situation. However, the sequential strategy described in the first interview may become an obstacle if the numbers are big and hinder the flexibility of thinking. Yet, the holistic relational strategy produced in the second interview addresses these issues. In our study, we observed multiple cases of students using the latter strategy when they followed EDA training.

6 DISCUSSION AND CONCLUSIONS

We suggested that in the case of solving simple additive word problems, sequential reasoning (addition and subtraction as processes) and holistic systemic reasoning (additive relationship between three quantities) complement each other. Some teaching approaches can generate a well-balanced development of both types of reasoning in pupils. In other cases, sequential reasoning can be overdeveloped in relation to systemic relational understanding, thus creating a reasoning disequilibrium. In this paper, we studied the effects of implementing the EDA within the Relational Paradigm on pupils' problem-solving success. The EDA explicitly requires students to model and analyze a problem as a system of quantitative relationships, thus equilibrating their sequential thinking by their holistic thinking. Our analyses of pupils' performance in the control and experimental groups suggest that different approaches to teaching additive word problem solving engender different patterns of success. The example of one student's strategies before and after EDA training illustrates the shift in students' reasoning. We discuss below some connections to other research and teaching practices.

Although many researchers highlighted the importance of relational thinking, the clear distinction between two paradigms and the idea of equilibrium provide an important theoretical improvement. By formulating the perspective on mathematical reasoning development in schools as the Relational Paradigm, our research continues the work of other researchers (Davydov, 2008; Iannece et al., 2009; Malara & Navarra, 2002; G. Zuckerman, 2004). Within this paradigm, we recognize the prevalence of relational thinking as a tool for problem solving and as an educational goal.

Based on our theoretical framework, we conjectured that the disequilibrium in thinking, with the dominance of sequential thinking, generates more difficulties for pupils in solving word problems requiring an inversion of semantic structure for their solution (*difficult* problems). Our data suggest the existence of such difficulties in the control group, which correlates with the situation observed by Steffe and Johnson (1971) and many other researchers (e.g., Nesher et al., 1982; Riley et al., 1984; Vergnaud, 1982). The control group in our study experienced the traditional teaching approach which promotes pupils' personal strategies in problem solving (e.g., Carpenter et al., 1993). The approach of promoting pupils' personal strategies can potentially promote the dominance of sequential reasoning in some pupils, making inconsistent problems difficult for them and thus creating a learning obstacle. It is also possible that the main cause of this negative effect is the traditional teaching approach developed within the Operational Paradigm, which focuses on arithmetic operations and calculation and gives little attention to structures and relationships.

Our results indirectly confirm the hypothesis of reasoning duality in additive word problem solving. We have started with this hypothesis in mind (Sfard, 1991) and have developed the EDA based on this hypothesis in the hope of producing a significant shift in pupils' thinking development. As our data shows the significant positive difference between the control and experimental groups, we can conclude that the reasoning duality hypothesis is indirectly confirmed.

Furthermore, our results support the idea of "the multidimensional landscape of developmental potentials" discussed by Zuckerman (2004) as an aspect of the *zone of proximal development* concept. Many researchers and practitioners see mathematical reasoning development as a linear process—where only one way is possible. Particularly in arithmetical problem solving, curricula proposes the path from counting and the calculation of simple operations, to solving simple problems. Russian researchers (e.g., Davydov, 2008; G. A. Zuckerman, 2004) designed and implemented a different curriculum path: from quantitative relationships of objects to abstract notation, to solving problems. We studied the possibility of parallel growth and the integration of both: the counting/calculation/operations and relational thinking lines of development. Our study shows that the idea of simultaneous development of different ways of thinking, while preserving an equilibrium and appropriate coordination between these ways of thinking, has a promising potential for teaching mathematics and for future research.

Our research contributes to the knowledge of problem-solving ability development. Similar to the theory of Realistic Mathematics Education, which focuses "on teaching the activity of mathematizing instead of teaching the results of the mathematizing activities of others" (Van Stiphout, Drijvers, & Gravemeijer, 2013), the EDA and Ethno-Mathematical model of problem solving focuses on teaching how to analyze and model situations described in natural language and how to connect the everyday story comprehension with relational mathematical thinking. However, specially designed mathematically unrealistic situations (written stories) in our study helped pupils to understand the additive relationship as an object, independently from their understanding of arithmetical operations and the real-life context.

Finally, relational thinking in mathematics is a precursor of algebraic reasoning (Kieran, 2014). Many researchers (e.g. Blanton et al., 2015; Cai & Knuth, 2011; Carraher & Martinez, 2008; Gerhard, 2009; Lins & Kaput, 2004) studied pupils' difficulties with algebra and argued for the development of algebraic reasoning in primary school. However, an early introduction of algebraic activities in the curricula is often disconnected from the arithmetical problem solving. The EDA experimented in our study connects algebraic and arithmetic activities and supports the equilibrium in reasoning development. A more detailed study is required to understand in detail how different teaching approaches contribute to supporting or precluding the relational reasoning and the equilibrium in pupils' reasoning development.

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