Multiplicative structures in elementary school mathematics: Relational approach

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In this theoretical essay, we critically analyze some multiplicative structures identified by researchers and practitioners in the field of teaching elementary school mathematics (grades 3-5). Drawing upon Davydov's theory of developmental instruction, we use the relational perspective and propose another classification and graphical representations of multiplicative structures. We suggest that the new approach may better support student understanding of multiplicative relationships and at the same time contribute to the foundation of their algebraic thinking.

INTRODUCTION

The teaching and learning of problem solving, a fundamental pillar of school education, always attracts researchers' close attention. Problem-solving activities formally debut at the beginning of schooling where students deal with simple word problems requiring one arithmetic operation for their solution. In order to support effective teaching and learning of problem solving at this stage, researchers analyzed simple word problems and their semantic structures and produced typologies of additive word problems (requiring one addition or subtraction operation) and multiplicative word problems (requiring one multiplication or division operation). However, theoretical clarification about the mathematics behind simple word problems did not lead to the elimination of teaching and learning difficulties. What's more, recent works about early algebra and modelling bring new perspective in the area of simple word problem solving.

In 2011-2014, responding to the practical needs of teachers in our region (Quebec), we conducted a research project focusing on additive problems and their teaching in grades 1 and 2. We aimed at facilitating students' learning to solve such problems as well as at developing their mathematical thinking. After having critically analyzed the literature concerning additive word problems, we proposed the existence of two distinct paradigms of research in the area: the operational paradigm and the relational paradigm. The operational paradigm recognizes arithmetic operations as the basis for understanding real world situations (or word problems) involving adding, removing, comparing, equalizing, sharing etc. The relational paradigm, however, preconizes the

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holistic and flexible understanding of simple relationships between three quantities as the foundation for solving such problems. Drawing upon Davydov's (1982) definition of the additive relationship as "the law of composition by which the relation between two elements determines a unique third element as a function" (p. 229), we tried to root the teaching and learning of additive problem solving in students' holistic and flexible understanding of this relationship. We also use a specific graphical representation of the additive relationship to allow students' modelling additive problems based on their understanding of the relationship involved. This new epistemological perspective proved to be fruitful allowing for the design of quite successful teaching strategies. The teachers who participated in the study reported that they could not imagine returning to the old way of teaching (based on operations and key words) (Savard et al., 2018).

Since 2016, an extension to our study has been aimed at examining the multiplicative word problem solving in grades 3-6. To date, following an in-depth analysis of available typologies of multiplicative situations (or word problems), their sematic structures, and their graphical representations, we analyzed the multiplicative situations from the relational perspective.

In this paper, we discuss simple multiplicative word problems, their structures and typologies, and their graphical representations as these are treated in the literature. We then discuss, in more detail, the relational perspective. We conclude by suggesting that specific representations of multiplicative structures, coherent with the relational paradigm, can enhance students' thinking mathematically and their learning to model and solve problems.

MULTIPLICATIVE STRUCTURES AND THEIR REPRESENTATIONS

Discussing understanding and representations in mathematical problem solving Vergnaud (1983) distinguishes two possible directions in their analysis: implicit representations the solver can have of the problem and explicit representations the solver might create to communicate the important elements and retained operations in form of graphic drawings and letter expressions. (Vergnaud, 1983, p. 33). Vergnaud suggested that the former informs the latter. Thus, researchers can observe and study students' graphical representations of problems in order to understand their implicit internal representations.

Some researchers (e.g. Rockwell, 2012; Mancl, 2011) also propose to explicitly teach schemes and graphical representations to students, especially to those having difficulties in mathematics, and claim that such practices facilitate problem solving. Some teachers' manuals (e.g. MEO, 2008; MELS, 2009) and official documents that guide teaching practices provide various typologies of multiplicative structures together with their graphical representations claiming that these representations should not only support teachers' understanding of these structures but also guide their teaching.

As part of our extended research, we examined these sources and analyzed the multiplicative structures and the representations provided in them in order to identify the kind of thinking they might potentially evoke in learners if used explicitly within the teaching-learning activity.

Some of the problem types (or categories) studied reflect mainly mathematical structures (e.g. Cartesian product, mapping rule, equal groups) while others pay special attention to actions (e.g., rate, multiplicative change, sharing), disposition (rectangular disposition), comparison expression (e.g., times more, times less) or what is unknown or invariant.

In addition, representations of multiplicative structures are also diverse and reflect different aspects of the represented situation. For example, Nunes, T., & Csapó, B. (2011) mention the following problem:

> "Together Rob and Ann have 15 books (quantity). Rob has twice the number of books that Ann has (or Ann has half the number of books that Rob has) (relation). How many books does each one have?" (p. 29)

This problem presents a multiplicative structure belonging to the multiplicative comparison category. The authors propose to represent this situation as shown on Figure (a).

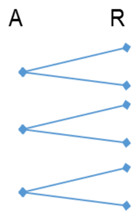
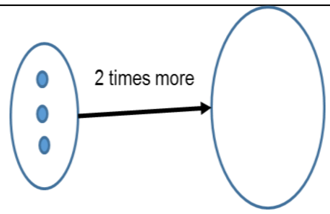
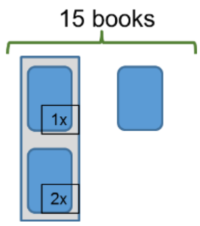
a) Adapted from Nunes, T., & Csapó, B. (2011).	b) Adapted from MELS (2009).	c) Adatpted from Van de Walle & Lovin, (2008)
		

Table 1: Representations of the book problem

A close look at this representation makes it clear that each point (extremities of arrows) in column A represents one of the books Ann has, and that each point in column R represents one of the books Rob has. The number each has, being unknown, makes it impossible to fully represent the situation.

Other sources propose different representations for the multiplicative comparison (b). In this representation, each element of the known set is represented by a dot and the comparison expression is shown as is in the problem (2 times more).

Using representations (a) and (b) to solve problems may become cumbersome if large numbers are involved or if the multiplicative structure is a part of a more complex structure, as in the book word problem above.

The representation (c) below, however, carries different characteristics. It shows repetitions of a set equal to the referred set and uses number notation to specify the comparison relationship. It does not show elements of singular sets. As such, it has an advantage over the first two representations of multiplicative structures because it can represent sets of any magnitude and can be combined with other representations to deal with more complex problems such as the book word problem described earlier.

Some other representations we found (e.g. MEO, 2008; Vergnaud,1983) *encode* the situation as an operation, using mathematical symbols or arbitrary icons. They do not represent objects, sets or relationships in *analog way* thus making it difficult for the observer to create a link of analogy between the problem and its representation.

The neuroscientist Yan Robertson (2017) explains that solver might ignore the essence of a problem by paying more attention to its superficial aspects. He explains that this behavior is quite natural at the beginning of the learning process, he also suggests the important role analogy and mental schema play in analyzing and solving a problem. He writes:

> It is obvious from the preceding discussion that analogies can play an important role in influencing thinking. One way that analogies can operate is by activating a schema that in turn influences the way we think about a current situation (p. 83).

The same, we argue, is true for representations: individuals represent the things they pay attention to and the way they understand it, using available knowledge and mental schemes. Thus, if the thinking about objects or about exact number of objects is what students pay attention to, they will probably try to use schemes a) or b). Once chosen by a solver, the type of representation shapes the entire thinking process. Trying to represent each object (e.g., each individual book), students will be blocked and unable to solve the problem. Representation (c), however, attracts solvers' attention directly to the relationship between sets; it is more flexible and can help to understand the entire problem at hand in its complexity.

We can conclude that some representations might lead to dead ends, thus becoming counterproductive in terms of learning because they do not support cases that are more complex. In some conditions, they can act as didactical obstacles preventing students from constructing of new representation of a problem. We propose that the explicit use of such limited and limiting representations by the teacher may evoke spurious elements in students' minds and eventually derail their attention from the multiplicative structure we would like them to recognize and be able to use to solve the problem.

In the context of Quebec, as well as in some other Provinces and countries around the world, it is generally accepted that students should develop their own representations to solve problems. However, speaking about "ratio" and "rate" Thompson (1994) argues:

> [H]ow one might classify a situation depends upon the operations by which one comprehends it. In Thompson (1989) I illustrate how an "objective" situation can be conceived in fundamentally different ways depending on quantitative operations available to and used by the person conceiving it (p. 16).

Indeed, according to Robertson (2017), many beginners turn their attention to irrelevant elements of the problem at hand, and therefore, the representations they may construct can lead them to dead ends. This means that allowing students to construct their own representations is not as effective a teaching strategy as many believe because students may fail to construct representations that reflect the mathematical relationship in the problem at hand. Based on Thompson's (1994) ideas, we suggest that for a student to construct a representation of a multiplicative structure (for example multiplicative comparison) he or she needs to possess the concept of this particular relationship to which we turn next.

RELATIONAL PERSPECTIVE

In his seminal work, Davydov (1982) put forward the idea of quantitative relationships as mathematical concepts that we need to teach and learn in elementary school, even prior to numbers. He argues that the concept of number appears from the multiplicative comparison of two magnitudes (or quantities), one playing the role of unit of measurement and the other being measured. In their experiments, researchers introduced students first with situations of comparing and measuring water, surfaces, and ropes through the manipulation of real objects rather than stories or word problems. They also used specific graphical and symbolic representations to analyze and discuss these situations with students. They then used the developed representations to support word-problem solving.

Based on Davydov's ideas, researchers in other countries successfully implemented the measuring approach in developing number concept with elementary students (e.g. Dougherty & Slovin, 2004). This growing research suggests that the relational approach might enhance students' mathematical thinking and particularly their understanding of unit (Barrett et al., 2011). As far as we know, the multiplicative relationships and their graphical representations did not yet attract adequate attention in the context of simple word problems. While graphical representations can be found in many textbooks and teacher guides, it is not yet clear how exactly teachers use these representations in classroom. Thus, the role these representations might play in the learning process is not theoretically developed.

According to the relational perspective, understanding a word problem means that students recognize the quantitative relationships involved. Sandra P. Marshall (1995) argues that it is possible to find the *basis set of schemas* (analogous to basis set of vectors in a space) to be able to understand and describe all and any multiplicative problem in a domain. What, then, can be a minimal set of multiplicative relationships allowing for the understanding of all multiplicative problems of the elementary school?

SIMPLE MULTIPLICATIVE RELATIONSHIPS AND THEIR REPRESENTATIONS

Studying works of Davydov, Thompson, and Vergnaud yields the identification of the following relationships.

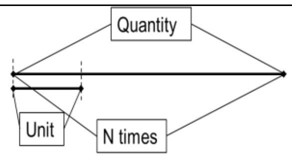
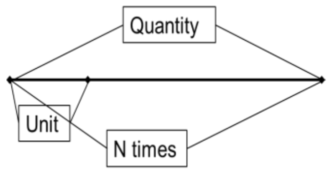
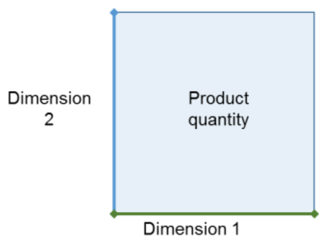
Relationship	Description	Representation	Examples
Multiplicative comparison or measurement	This relationship can be used if one quantity can be measured (or compared in multiplicative way) against another quantity that is physically distinct from the first one yielding a number whether it is known or unknown.		1. Max has three times as many marbles as Maya. 2. Max' shoe measures twice the Maya's shoe. 3. How much is Maya younger than Max?
Multiplicative composition	This relationship can be used if one quantity is composed of a number of equal parts (number can be rational as well).		4. Max has many boxes with the same number of marbles in each. 5. A car moving with a constant speed made a certain distance in a certain time.
Cartesian product	This relationship can be used if all three elements of the multiplicative relationship have different physical origins and none of them can be seen as a pure number or as a unit of measurement.		6. One uses a number of skirts and a number of blouses to create costumes. 7. One evaluates a rectangular area in relation to its length and width.

Table 2: Basic multiplicative relationships and their graphical representations.

Many pedagogical sources describe a proportion among multiplicative structures. Obviously, the proportional relationship deserves special attention. This relationship can be used when one quantity being measured by the second yields the same number as the third being measured by the fourth. The following example presents such a relationship.

> A glass of cocktail contains 40% of apple juice. For any two particular amounts of cocktail, the juice part measures the same (each time, we consider the amount of cocktail as a unit of measurement).

However, proportional relationship, in our view, is not a simple relationship so we do not discuss it here. This set of relationships is not like orthogonal basis vectors, because one can employ two or three of them (one at a time) to interpret the same simple word problem. However, this set is sufficient to interpret all multiplicative problems from the repertoire of elementary school. All graphical representations we propose show the underlying multiplicative relationship—thus highlighting the essential mathematical idea of the problem. At the same time, they do not show irrelevant elements such as objects or their numbers (these elements are irrelevant when we are looking for the arithmetic operation to figure out the unknown element of the relationship; they however, can be relevant when carrying out the chosen mathematical operation). The explicit use of these representations by the teacher may help not derail students' attention from the essence of the problem.

Representing a quantity by a segment or a lengthy rectangle has many advantages. First, there is a way to imagine objects arranged into a line. This mental organization allows for the representation of any number of objects as well as the preservation of the meaningful link between the initial situation and its representation. Second, it helps to represent an unknown quantity because it can be imagined as a line of objects. Finally, all elements of the relationship can be visually represented and simultaneously analyzed, which, in turn, helps to derive the arithmetic operation to find out the unknown element.

The representations we propose can be easily combined to describe more complex situations. As we mentioned above, many textbooks in some countries use visual representations to support students' problem solving. Yet, why and how they might help is not theoretically clear. We support Davydov's idea that the concept of multiplicative relationship(s) is to be taught and developed by students prior to solving complex word problems. Therefore, we believe that it is more effective to conceptualize the activity of modelling, discussing, and solving simple word problems through development of the concept of multiplicative relationship. This activity is not *problem solving* per se, because students should not just apply their knowledge of multiplication and division to find a solution to a problem. Rather, this activity is learning about basic relationships and developing mental schemes that will eventually support analysis, modelling, and solving of situations that are more complex, thus allowing for a real problem solving.

We would like to share our classroom experience in developing multiplicative relationship with students. Space limitation precludes a description of such work but we are looking forward to presenting these at the conference.

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